Integrability of characteristic Hamiltonian systems on simple Lie groups with standard Poisson Lie structure

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Abstract

Phase space of a characteristic Hamiltonian system is a symplectic leaf of a factorizable Poisson Lie group. Its Hamiltonian is a restriction to the symplectic leaf of a function on the group which is invariant with respect to conjugations. It is shown in this paper that such system is always integrable.

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1 Introduction

The discovery of Lax pair for the Korteveg-de-Vris equation opened new class of exactly solvable ordinary and partial differential equations. Solutions to such equations can be expressed in terms of solutions to certain spectral problems, or in terms of certain factorization problems of either Gauss type or Riemann-Hilbert type.

It has been discovered later that for most of such systems there is a Hamiltonian structure with respect to which they are integrable system in a sense of Lioville.

A Lie theoretical explanation of this fact has been provided by Kostant [K79] on the example of the Toda system. He noticed that the phase space of the Toda system can be naturally identified with an orbit of a Borel subgroup acting by adjoint representation a simple Lie group which passes through the proncipal nilptent element in the opposite Borel subalgebra. The Hamiltonians of the Toda system are restrictions of central functions on the corresponding simple Lie algebra to these orbits. Then Adler [Adl79] applied this approach to the KdV equation and interpreted it in terms of Lie algebra of pseudo-differential operators (see also [Sym80]). This approach has been generalized in [RSTS79]. The key observation was that one should consider certain Lie algebra structure on the space dual to the Lie algebra instead of Borel subalgebras.

It has been discovered in a number of examples [S80] that if the Poisson brackets between matrix elements of Lax operator have certain special structure (so-called r-matrix structure) and that spectral functions on such Lax operators generate integrable systems. In contrast

to Toda systems, KdV equations and others interpreted in terms of Lie algebras, most of these systems had nonlinear (quadratic) Poisson brackets. Such Poisson structures were later categorized by Drinfeld [Dr87] when he introduced the notion of a Poisson Lie group. The theory of Kostant was generalized to Poisson Lie groups by Semenov-Tian-Shanski [STS85]. In this case the phase space of a system is a symplectic leaf of a factorizable Poisson Lie group. Integrals of such system are adjoint invariant functions restricted to the symplectic leaf. They Poisson commute, but generically there will less independent invariant functions then half of the dimension of the phase space. However, the Hamiltonian flow generated by an adjoint invariant function can be explicitly described in terms of the factorization on corresponding Lie group, which is an indication of integrability of such systems. We will call such systems characteristic (in algebraic case the integrals are characters of finite dimensional representations).

As it was already mentioned that the phase space of the Toda system corresponding to a simple Lie algebra $\mathfrak{g}$ is a special coadjoint orbit of the Poisson Lie group whose tangent Lie bialgebra is dual to $\mathfrak{g}$. The integrals are given by restriction to this orbit of functions which are invariant with respect to the adjoint action of $\mathfrak{g}$. It is natural to study Hamiltonian systems generated by such functions on other orbits. This has been done in [DLNT86] [EFS93][GS97]. In [DLNT86] the integrability of such systems on generic orbit has been proven for classical Lie algebras. In [EFS93] it was argued that such systems are integrable for generic triangular orbits. In [GS97] it was proven that such systems on generic orbit are integrable for any simple finite dimensional Lie algebra $\mathfrak{g}$. An important result of [GS97] is that these systems are not only integrable in the usual Lioville sense but there are also so-called degenerate integrable systems (with the dimension of the invariant tori less then half of the dimension of the phase space). Such degenerate integrable systems were systematically introduced in [N72] and they are also known as systems with non-commutative integrability (see [F88] where they were used in a special case). This notion is a classical version of hidden symmetries in quantum mechanics (see [P26] [FMSUW65]).

Similar question can be asked about corresponding Poisson Lie groups. In this case special symplectic leaves give generalizations of Toda systems [HKKR] known for SL_n as relativistic Toda systems. For generic symplectic leaves of SL_n the integrability of characteristic systems has been proven in [Li97].

In this paper we will show the degenerate integrability of a characteristic Hamiltonian system on any symplectic leaf of any simple Poisson Lie groups. We will focus on complex algebraic case and will consider the systems corresponding to real forms only briefly.

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2 Characteristic Hamiltonian systems of factorizable Poisson Lie groups

2.1 Factorizable Lie bialgebras

Recall that a Lie bialgebra is a pair $(\mathfrak{g}, \delta)$ where $\mathfrak{g}$ is a Lie algebra and the linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a $\mathfrak{g} \wedge \mathfrak{g}$-valued 1-cocycle on $\mathfrak{g}$ which induces Lie algebra structure on the dual vector space to $\mathfrak{g}$.

A Lie bialgebra $(\mathfrak{g}, \delta)$is called factorizable if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that:

- $r + \sigma(r)$ $(\sigma(x \otimes y) = y \otimes x)$ is a nondegenerate element of $\mathfrak{g} \otimes \mathfrak{g}$ invariant with respect to the diagonal adjoint $\mathfrak{g}$-action,
- $\delta(x) = [r, x]$, where the bracket is the diagonal adjoint action of x on $\mathfrak{g} \otimes \mathfrak{g}$ and the result is in $\mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$,
- $$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

  where $\mathfrak{g}^{\otimes 3} \subset U(\mathfrak{g})^{\otimes 3}$ and $r_{12} = r \otimes 1, r_{23} = 1 \otimes r$ etc..

Let $\mathfrak{g}$ be a factorizable Lie bialgebra with classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.

Lemma 1 *The linear maps* $r_\pm : \mathfrak{g}^* \longrightarrow \mathfrak{g}$

$$r_+(l) := \langle r, l \otimes id \rangle, \quad r_-(l) := -\langle r, id \otimes l \rangle \qquad \forall\, l \in \mathfrak{g}^*;$$

are Lie bialgebra homomorphisms.

The proof of this lemma we will leave as an exercise. It follows from the classical Yang-Baxter equation for r.

The linear map $I : \mathfrak{g}^* \to \mathfrak{g},\ l \mapsto r_+(l) - r_-(l)$ is a linear isomorphism. It is called *the factorization map.*

Corollary 1 *The subspaces* $\mathfrak{g}_\pm = Im(r_\pm)$ *are Lie subbialgebras in the Lie bialgebra* $\mathfrak{g}$.

Lemma 2 (1) *The subspaces* $\mathfrak{n}_\pm = r_\pm(ker(r_\mp))$ *are Lie ideals in* $\mathfrak{g}_\pm$ *respectively.*

(2) *The map* $\theta : \mathfrak{g}_+/\mathfrak{n}_+ \to \mathfrak{g}_-/\mathfrak{n}_-$ *which sends the residue class of* $r_+(e) mod\ \mathfrak{n}_+$ *to* $r_-(e) mod\ \mathfrak{n}_-$ *is defined and is an isomorphism of Lie algebras.*

Proof. The first statement follows from the facts that $\ker(r_\pm)$ are Lie ideals in $\mathfrak{g}^*$ and that $r_\pm$ are Lie algebra homomorphisms.

Let $\ell \in \mathfrak{g}^*$, $n \in \ker(r_+)$, $m \in \ker(r_-)$. Consider representative $r_+(\ell) + r_+(m)$ of $r_+(\ell)\mathrm{mod}(\mathfrak{n}_+)$ by linearity $r_+(\ell)+r_+(m) = r_+(\ell+m) = r_+(\ell+n+m)$. Now, $r_-(n+\ell+m) = r_-(\ell+n) = r_-(\ell)+r_-(n)$ represents the equivalence class $r_-(\ell)\mathrm{mod}(\mathfrak{n}_-)$. Therefore the map $\theta : \mathfrak{g}_+/\mathfrak{n}_+ \to \mathfrak{g}_-/\mathfrak{n}_-$ is defined and it is a linear isomorphism.

We leave as an exercise to prove that it is a Lie algebra homomorphism.

Theorem 1 (1) *Every element $x \in \mathfrak{g}$ admits unique factorization*

$$x = x_+ - x_-$$

where $x_\pm \in \mathfrak{g}_\pm$, and $\theta(x_+ \mathit{mod}\ \mathfrak{n}_+) = x_- \mathit{mod}\ \mathfrak{n}_-$.

(2) *The Lie algebra $\mathfrak{g}^*$ is isomorphic to the following Lie subalgebra of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$:*

$$\{(x_+, x_-) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \mid \theta(x_+ \mathit{mod}\ \mathfrak{n}_+) = x_- \mathit{mod}\ \mathfrak{n}_-\}$$

(3) *If we model $\mathfrak{g}^*$ as in* (2) *the factorization map $I : x \mapsto r_+(x) - r_-(x)$ acts as $(x_+, x_-) \mapsto x_+ - x_-$, where on the right side we consider $\mathfrak{g}_\pm$ as Lie subalgebras of $\mathfrak{g}$.*

2.2 Factorizable Poisson Lie groups

A Poisson Lie group is a Lie group with the such Poison manifold structure on it that the multiplication map $G \times G \to G$ is a Poisson map.

There is a bijection between connected simply-connected finite- dimensional Poisson Lie groups and finite-dimensional Lie bialgebras. Each finite-dimensional Lie bialgebra can be "exponentiated" to a connected simply-connected Poisson Lie group and conversely, a Poisson Lie structure on a finite-dimensional Poisson Lie group defines a Lie bialgebra structure on the corresponding Lie algebra. This Lie bialgebra structure is called tangent Lie bialgebra to a Poisson Lie group. Thus, we have Poisson Lie counterparts of Lie bialgebras introduced above: coboundary, quasitriangular, triangular, factorizable.

If (G, p) is a quasitriangular Poisson Lie group, the Poisson tensor has the following explicit description:

$$p(x) = \mathrm{Ad}_x(r) - r \in \wedge^2 TG \simeq \wedge^2 g\ .$$

Here we trivialized the tangent bundle by left translations. For the Poisson brackets on a quasitriangular Poisson Lie group we have:

$$\{f_1, f_2\} = \langle r, d_l f_1 \wedge d_l f_2 \rangle - \langle r, d_r f_1 \wedge d_r f_2 \rangle$$

where d_l and d_r are, respectively, left and right differentials on G.

For factorizable Poisson Lie groups we have

- maps $r_\pm$ lift to Lie group homomorphisms $r_\pm : G^* \to G$.
- $G^\pm = \mathrm{Im}(r_\pm) \subset G$ are Poisson Lie subgroups (connected simply connected)
- $N^\pm = \mathrm{Im}(\ker(r_\pm)) \subset G^\pm$ are normal Lie subgroups
- Lie algebra isomorphism $\theta : \mathfrak{g}_+/\mathfrak{n}^+ \xrightarrow{\sim} \mathfrak{g}_-/\mathfrak{n}_-$ lifts to Lie group isomorphism $\theta : G^+/N^+ \xrightarrow{\sim} G^-/N^-$.
- Lie group G^* can be modeled as:

$$G^* = \{(g_+, g_-) \in G^+ \times G^- \mid \theta(g_+ \mathrm{mod}\ N^+) = g_- \mathrm{mod}\ N^-\}$$

- There exist open dense subsets $G', G'' \subset G$ such that for each $g \in G'$ there exists unique factorization $g = g_+ g_-^{-1}$, $\theta(g_+ \mathrm{mod}\ N_+) = g_- \mathrm{mod}\ N^-$ and for each $g \in G''$ there exists unique factorization $g = g_-^{-1} g_+$ with the same conditions on $g_\pm$.
- Left, respectively right, *factorization maps* $G^* \to G$ map (g_+, g_-) to $(g_+ g_-^{-1})$, respectively to $g_-^{-1} g_+$.

Here we assume that G^* is represented as a subgroup of $G^+ \times G^-$.

2.3 The double

The double $D(\mathfrak{g})$ of the Lie bialgebra $\mathfrak{g}$ is the Lie bialgebra which is the direct sum $\mathfrak{g} \oplus \mathfrak{g}^{*op}$ of Lie coalgebras. The Lie bracket on it is determined uniquely by the requirement that the natural bilinear form $< (x, l), (y, m) >= l(y) + m(x)$ is $D(\mathfrak{g})$-invariant and the isotropic subspaces $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras. We will denote these Lie bialgebra inclusions $i : \mathfrak{g} \to D(\mathfrak{g})$ and $j : \mathfrak{g}^{*op} \to D(\mathfrak{g})$.

The double $D(G)$ of G is the connected, simply connected Poisson Lie group having $D(\mathfrak{g})$ as its Lie bialgebra. The maps i and j lift to injective Poisson Lie maps $i : G \to D(G)$, $j : G^{*op} \to D(G)$ and consequently to a map $\mu \circ (i \times j) : G \times G^{*op} \to D(G)$: $(x, y) \mapsto i(x)j(y)$ which is also a local Poisson isomorphism. Here G^{*op} is the Lie Poisson group G^* with the opposite Poisson structure.

2.4 Symplectic leaves in Poisson Lie groups

The Poisson Lie group G^{*op} acts on $D(G)$ via left multiplication, $y \cdot x := j(y)x$. We also have a map $\varphi : G \to D(G)/j(G^{*op})$ which is the composition of i with the natural projection. This map is Poisson. In a neighborhood of the identity this map φ is a Poisson isomorphism. The map φ has open dense range but it is not surgective if the factorization problem in

$D(G)$ does not have global solution. It intertwines local dressing action of G^* on G [STS85] and the action of G^* on the cosets.

The symplectic leaves of G are orbits of dressing action of G^{*op}. Or, equivalently, they are connected components of preimages of left G^{*op}-orbits in $D(G)/j(G^{*op})$. Notice that this description does use the Poisson structure on the groups and therefore we can use notation G^* instead of G^{*op} without a danger of confusion.

2.5 Characteristic Hamiltonian systems on factorizable Poisson Lie groups

Let (G, p) be a factorizable Poisson-Lie group. Let $I(G) \subset C^\infty(G)$ be the subspace of Ad_G-invariant functions on G.

Theorem 2

i) $I(G)$ is a commutative Poisson algebra in $C^\infty(G)$.

ii) In a neighborhood of $t = 0$ the flow lines of the Hamiltonian flow induced by $H \in I(G)$ passing through $x \in G$ at $t = 0$ have the form

$$x(t) = g_\pm(t)^{-1} x g_\pm(t),$$

where the mappings $g_\pm(t)$ are determined by

$$g_+(t) g_-(t)^{-1} = \exp\left(tI\left(dH(x)\right)\right),$$

and $I : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is the factorization isomorphism.

Definition 1 *Characteristic Hamiltonian system on a factorizable Poisson Lie group is a Hamiltonian system whose phase space is a symplectic leaf of a factorizable Poisson Lie group and whose Hamiltonian is an adjoint invariant function on G.*

The theorem above implies that the equations of motion of a characteristic Hamiltonian system on a factorizable Poisson Lie group can be solved via factorization. Below we will show that such systems are integrable for all simple Poisson Lie groups.

3 Standard Poisson Lie structure on simple Lie groups

3.1 Standard factorizable Lie bi-algebra structure on simple Lie algebras and standard Poisson Lie structure on simple Lie groups

Let $\mathfrak{g}$ be a simple complex Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$. Let (H_i, e_i, f_i) $i = 1, \ldots, r = \mathrm{rank}(\mathfrak{g})$ be elements of the Chevalley basis of $\mathfrak{g}$ for this choice of Borel subalgebra which correspond to simple roots. It is well known that $\mathfrak{g}$ is freely generated by (H_i, e_i, f_i) modulo determining relations

$$\begin{aligned} [H_i, H_j] &= 0, && (1)\\ [H_i, e_j] &= a_{ij} e_j, && (2)\\ [H_i, f_j] &= -a_{ij} f_j, && (3)\\ [e_i, f_j] &= \delta_{ij} H_i, && (4)\\ (ad_{e_i})^{1-a_{ij}}(e_j) &= 0, \qquad i \neq j, && (5)\\ (ad_{f_i})^{1-a_{ij}}(f_j) &= 0, \qquad i \neq j, && (6) \end{aligned}$$

where (a_{ij}) denotes the Cartan matrix of $\mathfrak{g}$.

Consider a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ acting on the generators as

$$\begin{aligned} \delta(H_i) &= 0, && (7)\\ \delta(e_i) &= \frac{1}{2} d_i H_i \wedge e_i, && (8)\\ \delta(f_i) &= \frac{1}{2} d_i H_i \wedge f_i, && (9) \end{aligned}$$

where d_i is the length of the i-th root, in particular,

$$d_i a_{ij} = a_{ji} d_j.$$

Theorem 3 *There exists unique such linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ which has the following properties:*

i) δ is a 1-cocycle.

ii) $(\delta \wedge id) \circ \delta = 0$.

Remark 1 The definition of the standard Lie bialgebra structure on $\mathfrak{g}$ requires the choice of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Theorem 4 *$(\mathfrak{g}, \delta)$ is factorizable with*

$$r = \frac{1}{2} \sum_{ij} (B^{-1})_{ij} H_i \otimes H_j + \sum_{a>0} e_\alpha \otimes f_\alpha,$$

where $B_{ij} = d_i a_{ij}$ is the symmetrized Cartan matrix.

This induces the Poisson Lie structure on G for which the Lie bialgebra described above is the tangent Lie bialgebra. The Borel subgroup B and its opposite B^- are Poisson Lie subgroups.

The Lie bialgebra $Lie(G)$ is isomorphic to the double of the Lie bialgebra $Lie(B)$ quotioned by the diagonally embedded Cartan subalgebra [Dr87].

We denote by $\mathfrak{n}^+$ and N^- the nilpotent subgroups of B^- and B^-, respectively. Since $H = B^+ \cap B^-$ we have two natural projections $[\ ]_0 : B^+ \to B^+/N^+ \cong H$ $b \mapsto [b]_0$ and $[\ ]_0 : B^- \to B^-/N^- \cong H$. We shall also write B^+ and N^+ for B and N, respectively.

3.2 Symplectic leaves

3.2.1 Symplectic leaves of $B^\pm$.

It is known that $(B^+)^{*\mathrm{op}} \simeq B^-$ as a Poisson Lie group and that $D(B^+) \simeq G \times H$ as a Lie group. Fix these isomorphisms. The double $D(B^+)$ is a factorizable Poisson Lie group with Poisson Lie imbeddings $i : B^+ \hookrightarrow D(B^+)$, $j : B^- \hookrightarrow D(B^+)$

$$i(b^+) = (b^+, [b^+]_0) \ , \qquad j(b^-) = (b^-, [b^-]_0^{-1})$$

The group B^- acts on cosets $D(B^+)/j(B^-)$ by multiplication from the left. Define Lie subalgebras $\mathfrak{h}^w = coker\{w - id\}$ and $\mathfrak{h}_w = ker\{w - id\}$ of the Cartan subalgebra. Here the element $w \in W$ of the Weyl group W of G is considered as a linear operator on the Cartan subalgebra. Let H^w and H_w be corresponding Lie subgroups in H.

We have left Bruhat decoposition of $D(B^+)$:

$$D(B^+) = \sqcup_{w \in W} D(B^+)_w, \quad D(B^+)_w = B^- w B^- \times H.$$

Orbits of the action of B^- on the cosets $D(B^+)/j(B^-)$ have the following structure:

- $j(B^-) \backslash D(B^+)_w / j(B^-) \cong H_w$
- each B^- orbit in $D(B^+)_w/j(B^-)$ is isomorphic to $N_w^- \times H^w$.

Here N_w^- is the subspace of the nilpotent subgroup N^- generated by one parametric subgroups generated by those negative roots which remain negative after the action of w.

Symplectic leaves of B^+ are irreducible components of preimages of B^--orbits with respect to the map

$$\varphi : B^+ \stackrel{i}{\hookrightarrow} D(B^+) \longrightarrow D(B^+)/j(B^-) \ .$$

The image of φ intersects $j(B^-)$ orbits. Consider sets $B_w^+ = B^+ \cap B^- w B^-$. It is a Poisson subvariety [DCKP92] [HKKR] in G.

Lemma 3 *Let $\mathcal{O}_w \subset B^- w B^- \times H/j(B^-)$ be an orbit of the right B^- action on this coset, then $\varphi(B_w^+) \cap \mathcal{O}_w \subset \mathcal{O}_w$ is Zariski open.*

From this and from the fact that φ is a cover map one can show that $B_w^+ = B^+ \cap B^- w B^-$ is a Poisson subvariety in B^+ with symplectic leaves of have dimension $\ell(w)+$ corank $(w-\mathrm{id})$ [DCKP92].

One can give "explicit" description of symplectic leaves as connected components of Casimir functions.

According to [FZ99] define generalized minors as the following functions on the group G. Let G_0 be the subset in G formed by elements who have Gaussian factorization $x = [x]_-[x]_0[x]_+$ with $[x]_\pm \in N^\pm$ and $[x]_0 \in H$. For a weight λ define function

$$\Delta_\lambda(x) = [x]_0^\lambda$$

.

Let ω_i be a highest weight of i-th fundamental representation of G and $\bar{\bar{u}}$ and $\bar{v}$ are special representatives of elements $u, v \in W$ in G. Generalized minors are the following functions:

$$\Delta_{u\omega_i, v\omega_i} = \Delta_{\omega_i}(\bar{\bar{u}}^{-1} x \bar{v}) \tag{10}$$

where $\bar{u}$ and $\bar{\bar{u}}$ are special representatives of the element $u \in W$ (see [FZ99] for details).

Lemma 4 *The generalized minors $\Delta_{\omega_i, w^{-1}\omega_i}(x)$ do not vanish on B_w^+ and generalized minors $\Delta_{w\omega_i, \omega_i}(x)$ do not vanish on B_w^-.*

Proof. Let us prove that $\Delta_{\omega_i, w^{-1}\omega_i}$ does not vanish on $B^- w B^-$. Consider $x = b_- \overline{w} \beta_- \in B^- w B^-$ and let $b_- = [b_-]_0 [b_-]_-$ and $\beta_- = [\beta_-]_0 [\beta_-]_-$ be Gauss decompositions, then we have

$$\begin{aligned} \Delta_{\omega_i, w^{-1}\omega_i}(b_- \overline{w} \beta_-) &= \Delta_{\omega_i}([b_-]_0 \overline{w} \beta_- \overline{w}^{-1}) = \\ \Delta_{\omega_i, \omega_i}([b_-]_0 w([\beta_-]_0) \overline{w} [\beta_-]_- \overline{w}^{-1}) &= \Delta_{\omega_i}([b_-]_0 w([\beta_-]_0)) \end{aligned}$$

The last function does not vanish, which proves the first statement of the lemma. The proof of the last one is similar.

Here and below we will denote by $\Lambda = \mathbb{Z}\omega_1 \oplus \ldots \mathbb{Z}\omega_r$ the weight lattice in $\mathfrak{h}^*$.

Proposition 1 *Symplectic leaves of B_w^+ are irreducible components of level surfaces of functions*

$$c_{w,t}^+(x) = \prod_{i=1}^r \Delta_{\omega_i,w^{-1}\omega_i}(x)^{t_i}([x]_0^{\omega_i})^{t_i} \tag{11}$$

where $t = \sum_{i=1}^r t_i\omega_i \in \ker_\Lambda(w - \mathrm{id}) \in \mathfrak{h}^*$.

Proof. First, let us prove that functions $c_{w,t}^+(x)$ are invariant with respect to the (local) dressing action.

Dressing action of $b_- \in B^-$ on $b^+ \in B^+$ is the map $b_- : b_+ \mapsto b_+^{b_-}$ given by the solution to the factorization equations:

$$\begin{aligned} b_+ b_-^{-1} &= (b_-^{b_+})^{-1} b_+^{b_-} \\ [b_+]_0 \ [b_-]_0 &= [(b_-^{b_+}]_0 \ [(b_-^{b_-}]_0 \end{aligned}$$

where $b_\mp^{b_\pm} \in B^\mp$. This system has a unique solution when b_- is sufficiently close to 1.

1. Assume that $[b_-]_- = 1$, then $b_- = [b_-]_0 \in H$ and

$$b_+^{b_-} = [b_-]_0 \ b_+ [b_-]_0^{-1} \ .$$

It is clear that functions $c_{u,t}^+$ are invariant with respect to such action of $H \subset B^-$ iff $t = u(t)$.

2. Assume that $[b_-]_0 = 1$, then $b_- \in N^-$. Denote by $\tilde{x}_\pm \in B^\pm$ the result of "opposite" factorization of $(x, h) \in G \times H = D(B^+)$:

$$\tilde{x}_-^{-1}\tilde{x}_+ = x \ , \qquad [\tilde{x}_-]_0 \ [\tilde{x}_+]_0 = h \ .$$

Then we have

$$\begin{aligned} b_+ b_-^{-1} &= b_-^{-1}(b_- b_+ b_-^{-1}) = b_-^{-1}(\widetilde{b_- b_+ b_-^{-1}})_-^{-1} \ (\widetilde{b_- b_+ b_-^{-1}})_+ \\ [b_+]_0 &= [(\widetilde{b_- b_+ b_-^{-1}})_-]_0 \ [(\widetilde{b_- b_+ b_-^{-1}})_+]_0 \end{aligned} \tag{12}$$

Lemma 5 *Functions $\Delta_{\omega_i,u^{-1}\omega_i}(x)$ on $B_- u B_- \subset G$ are invariant with respect to the left action of N^-.*

Proof. Let $x \in B_- u B_-$ and $x = \beta'_- \bar{u} \beta''_-$ where $\bar{u} \in G$ is a special representative of $u \in W$ (see FZ for definition of $\bar{u}$), $\beta'_- \in N^-$ and $\beta''_- \in B^-$. Then for $n_- \in N^-$ we have

$$\begin{aligned} \Delta_{\omega_i,u^{-1}\omega_i}(xn_-) &= \Delta_{\omega_i}(xn_-\bar{u}^{-1}) = \Delta_{\omega_i}(\beta'_-\bar{u}\beta''_-n_-\bar{u}^{-1}) \\ &= \Delta_{\omega_i}(u([\beta''_-]_0)\bar{u}\beta''_-n_-\bar{u}^{-1}) \end{aligned}$$

The element $\bar{u}\beta''_- n_- \bar{u}^{-1}$ always admits factorization into the product $x_- x_+, \;\; x_\mp \in N^\mp$ and therefore

$$\Delta_{\omega_i, u^{-1}\omega_i}(xn_-) = \Delta_{\omega_i}(u[\beta'']_0) = \Delta_{\omega_i, u^{-1}\omega_i}(x) \; .$$

Now let us compute $\Delta_{\omega_i, u^{-1}\omega_i}(b_+^{b_-})$ for $b_- \in N^-$:

$$\begin{aligned}
\Delta_{\omega_i, u^{-1}\omega_i}(b_+^{b_-}) &= \Delta_{\omega_i, u^{-1}\omega_i}(\widetilde{b_- b_+ b_-^{-1}})_+ \\
&= \Delta_{\omega_i, u^{-1}\omega_i}(b_- b_+ b_-^{-1})([b_- b_+ b_-^{-1})_-]_0^{\omega_i})^{-1} \\
&= [b_+]_0^{\omega_i}([(\widetilde{b_- b_+ b_-^{-1}})_+]_0^{\omega_i})^{-1} \Delta_{\omega_i, u^{-1}\omega_i}(b_+ b_-^{-1}) \\
&= [b_+]_0^{\omega_i}([b_+^{b_-}]_0^{\omega_i})^{-1} \Delta_{\omega_i, u^{-1}\omega_i}(b_+) \; .
\end{aligned}$$

Here we used (12) (factor b) and the Lemma. Thus, the function

$$\Delta_{\omega_i, u^{-1}\omega_i}(b_+) \; [b_+]_0^{\omega_i}$$

is invariant with respect to the dressing action of N^-.

Thus, functions (11) are invariant with respect to the action of H and N^- and therefore they are invariant with respect to the dressing action of B^- on $B^- u B^-$ and therefore they are Poisson Casimirs for $B^- u B^-$. They do not vanish on $B^- u B^-$ and therefore their level surfaces are Poisson subvarieties and they form a fiber bundle over $(\mathbb{C}^\times)^{\mathrm{corank}(u - \mathrm{id})}$. Dimension of fibers coincide with the dimension of symplectic leaves of B^+ which are in $B^+ \cap B^- u B^-$ which proves the proposition.

Similarly for B^-. The subsets $B_w^- = B^- \cap B^+ w b^+$ are Poisson subvarieties whose symplectic leaves have dimension $\ell(w)+$ corank$(w - \mathrm{id})$.

Proposition 2 *Symplectic leaves of B_w^- are irreducible components of level surfaces of functions*

$$c_{w,s}^-(x) = \prod_{i=1}^r \Delta_{w\omega_i, \omega_i}(x)^{-s_i}([x]_0^{\omega_i})^{s_i}$$

where $s = \sum_{i=1}^r s_i \omega_i \in \ker_\Lambda (w - \mathrm{id}) \subset \mathfrak{h}^*$.

3.2.2 Symplectic leaves of $D(B^+)$.

The dual Poisson Lie group to $D(B^+) = G \times H$ can be identified with $B^+ \times B^-$ (as a Lie group). We also have an isomorphism of Lie groups $D(D(B^+)) \simeq D(B^+) \times D(B^+)$. Fix these isomorphisms.

The Poisson Lie imbeddings $i : D(B^+) \hookrightarrow D(B^+) \times D(B^+)$, $j : D(B_+)^{*\mathrm{op}} \hookrightarrow D(B^+) \times D(B^+)$ are

$$i(g,h) = ((g,h),(g,h)), \quad j((b_+,b_-)) = ((b_+,[b_+]_0),(b_-,[b_-]_0^{-1}) \ .$$

Symplectic leaves of $D(B^+) = G \times H$ are connected components of preimages of left $j(D(B^+)^{*\mathrm{op}}$-orbits in $D(B^+) \times D(B^+)/jD(B^+)^{*\mathrm{op}}$ with respect to the map

$$\phi : D(B^+) \hookrightarrow D(B^+)^{\times 2} \longrightarrow D(B^+)^{\times 2}/j(D(B^+)^{*\mathrm{op}}) \ .$$

Let $G^{u,v} = B^+uB^+ \cap B^-vB^-$ be the double Bruhat cell corresponding to the pair $(u,v) \in W \times W$. One can show that $G^{u,v} \times H$ is a Poisson subvariety with symplectic leaves of dimension $\ell(u) + \ell(v) + \mathrm{corank}(u - \mathrm{id}) + \mathrm{corank}(v - \mathrm{id})$ [HKKR]. Since generalized minors $\Delta_{\omega_i,u^{-1}\omega_i}(x)$ do not vanish on B_u^+ and generalized minors $\Delta_{v\omega_i,\omega_i}(x)$ do not vanish on B_v^- neither of them vanish on the intersection $G^{u,v} = B^+uB^+ \cap B^-vB^-$.

Next proposition describes symplectic leaves of $D(B^+)$ in terms of level sets of Casimir functions.

Proposition 3 *Symplectic leaves of $G^{u,v} \times H$ are irreducible components of level surfaces of functions*

$$c_{u,v,s,t}(x,h) = c^+_{u,s}(x,h)c^-_{v,t}(x,h)$$

where $s = \sum_{i=1}^t s_i\omega_i \in ker_\Lambda(u - id) \subset \mathfrak{h}^*$, $t = \sum_{i=1}^r t_i\omega_i \in ker_\Lambda(v - id) \subset \mathfrak{h}^*$ *and*

$$\begin{aligned} c^+_{u,s}(x,h) &= \prod_{i=1}^r \Delta_{\omega_i,u^{-1}\omega_i}(x)^{s_i}(h^{\omega_i})^{s_i} \ , \\ c^-_{v,t}(x,h) &= \prod_{i=1}^r \Delta_{v\omega_i,\omega_i}(x)^{t_i}(h^{\omega_i})^{-t_i} \ . \end{aligned}$$

Proof. We have the following natural identifications of groups:

$$D(B^+) = G \times H \ , \qquad D(B^+)^* = B^+ \times B^- \ , \qquad D(D(B^+)) = D(B^+) \times D(B^+) \ .$$

Left and right factorizations of an element in $D(D(B^+))$ are:

$$((g_1,h_1),(g_2,h_2)) = ((g,h)(g,h))(i(\xi_+),j(\xi_-)) = (i(\tilde{\xi}_+),j(\tilde{\xi}_-))((\tilde{g},\tilde{h}),(\tilde{g},\tilde{h})) \ ,$$

or, in components:

$$g\xi_\pm = \tilde{\xi}_\pm\tilde{g} \ , \qquad h[\xi_+]_0 = [\tilde{\xi}_\pm]_0^{\pm 1}\tilde{h} \ .$$

Such factorization exists on an open dense subset of $D(D(B))$.

The map $\xi : D(B^+) \to D(B^+)$ acting as $\xi : g \mapsto \tilde{g},\ h \to \tilde{h}$ determines the (local) dressing action. For each (g, h), it is defined for ξ sufficiently close to 1.

Using Lemma 5 we obtain

$$\Delta_{\omega_i, u^{-1}\omega_i}(\tilde{g}) = \Delta_{\omega_i, u^{-1}\omega_i}(\tilde{\xi}_-^{-1} g \xi_-) = [\tilde{\xi}_-]_0^{-\omega_i} [\xi_-]_0^{u^{-1}(\omega_i)} \Delta_{\omega_i, u^{-1}\omega_i}(g) \ .$$

Similarly,

$$\Delta_{v\omega_i, \omega_i}(\tilde{g}) = \Delta_{v\omega_i, \omega_i}(\tilde{\xi}_-^{-1} g \xi_+) = [\xi_+]_0^{\omega_i} [\tilde{\xi}_+]_0^{-v^{-1}(\omega_i)} \Delta_{v\omega_i, \omega_i}(g) \ .$$

From here we see that

$$\begin{aligned} &\prod_{i,j=1}^{r} \Delta_{\omega_i, u^{-1}\omega_i}(\tilde{g})^{t_i} \Delta_{v\omega_j, \omega_j}(\tilde{g})^{s_j} \\ &\quad = \prod_{i,j=1}^{r} \Delta_{\omega_i, u^{-1}\omega_i}(g)^{t_i} \Delta_{v\omega_j, \omega_j}(g)^{s_j} \\ &\quad \cdot [\xi_-]_0^{-t+u^{-1}(t)} [\xi_+]_0^{-v^{-1}(s)+s} (h\tilde{h}^{-1})^{-v^{-1}(s)+t} \end{aligned}$$

Thus, the functions $c_{u,v,t,s}$ on $D(B^+)$ are invariant with respect to dressing transformations if and only if

$$t = u(t) \ , \qquad s = v(s) \ .$$

Thus, functions $c_{u,v,t,s}$ are Casimirs in the Poisson algebras of functions on $G^{u,v} \times H$. On the other hand, they do not vanish on $G^{u,v} \times H$. Therefore their level surfaces form a fiber bundle over the torus $(\mathbb{C}^\times)^{\mathrm{corank}(uv^{-1}-\mathrm{id})} \times (\mathbb{C}^\times)^{\mathrm{corank}(u-\mathrm{id})}$. The fibers are Poisson subvarieties and they have the same dimension as symplectic leaves of $G^{u,v} \times H$.

3.2.3 Symplectic leaves of G.

Since G is a factorizable Poisson Lie group and its symplectic leaves can be described very similarly to those of $D(B^+)$ [HL93]. Taking into account that $D(G) = G \times G$ we have the composition of Poisson maps:

$$\phi : G \hookrightarrow G \times G \to (G \times G)/j(G^*)$$

where $j(G^*) = \{(b, b_-) \in B^+ \times B^- | [b] = [b_-]^{-1}\}$. Connected components of primages of $j(G^*)$-orbits on cosets are symplectic leaves of G.

The following proposition describes symplectic leaves of G in terms of Casimir functions.

Proposition 4 *Double Bruhat cell* $G^{u,v} = B^+uB^+ \cap B^-vB^-$ *is a Poisson subvariety in* G *with symplectic leaves of dimension* $\ell(u) + \ell(v) + \mathrm{corank}(uv^{-1} - \mathrm{id})$. *They are irreducible components of level surfaces of functions*

$$c_{u,v,t}(x) = \prod_{i=1}^{r} \Delta_{v\omega_i,\omega_i}(x)^{t_i} \Delta_{\omega_i,u^{-1}\omega_i}(x)^{u^{-1}(t)_i}$$

where $t = \sum_{i=1}^r t_i\omega_i \in \ker_\Lambda(uv^{-1} - \mathrm{id}) \subset \mathfrak{h}^*$.

Proof. The (local) dressing action of $G^* = \{(\xi_+, \xi_-) \mid [\xi_+]_0 = [\xi_-]_0^{-1}\} \subset B^+ \times B^-$ on G is given by the map $(\xi_+, \xi_-) : g \mapsto \tilde{g}$, where $\tilde{g}$ is the solution to the factorization problem:

$$\begin{aligned} g\xi_\pm &= \tilde{\xi}_\pm \tilde{g} \\ [\xi_+]_0 &= [\xi_-]_0^{-1} , \qquad [\tilde{\xi}_+]_0 = [\tilde{\xi}_-]_0^{-1} . \end{aligned} \tag{13}$$

Such $\tilde{g}, \tilde{\xi}_\pm$ exist for each g where (ξ_+, ξ_-) are sufficiently close to 1. Double Bruhat cells are invariant submanifolds for this action.

On $G^{u,v}$ we have (as above for $D(B^+)$)

$$\begin{aligned} \Delta_{\omega_i,u^{-1}\omega_i}(\tilde{g}) &= [\tilde{\xi}_-]_0^{-\omega_i} [\xi_-]_0^{u^{-1}(\omega_i)} \Delta_{\omega_i,u^{-1}\omega_i}(g) \\ \Delta_{v\omega_\ell,\omega_\ell}(\tilde{g}) &= [\xi_+]_0^{\omega_i} [\tilde{\xi}_+]_0^{-v^{-1}(\omega_i)} \Delta_{v\omega_i,\omega_i}(g) \end{aligned}$$

Therefore

$$\prod_{ij=1}^{r} \Delta_{\omega_i,u^{-1}\omega_i}(\tilde{g})^{t_i} \Delta_{v\omega_j,\omega_j}(\tilde{g})^{s_j} \tag{14}$$

$$= \prod_{ij=1}^{r} \Delta_{\omega_i,u^{-1}\omega_i}(g)^{t_i} \Delta_{v\omega_j,\omega_j}(g)^{s_j} \tag{15}$$

$$\cdot [\tilde{\xi}_-]_0^{-t} [\xi_-]_0^{u^{-1}(t)} [\tilde{\xi}_+]_0^{-v^{-1}(s)} [\xi_+]_0^{s} \tag{16}$$

Using relations (13) we can write the "ξ"-factor as

$$[\tilde{\xi}_-]_0^{-t+v(s)} [\xi_-]_0^{u^{-1}(t)-s} .$$

Thus, functions (14) are invariant if and only if

$$t = (s) , \qquad s = u^{-1}(t) .$$

Thus, the functions (14) are Poisson Casimirs and they do not vanish on $G^{u,v}$. Therefore their level sets form the fiber bundle over

$$(\mathbb{C}^\times)^{\mathrm{corank}(uv^{-1}-\mathrm{id})}$$

with fibers being Poisson submanifolds. The dimension of the fibers is the same as of the symplectic leaves of $G^{u,v}$. This proves the theorem.

4 Degenerate integrability of Hamiltonian systems

The notion of degenerate integrability was introduced in [N72]. First examples of such systems were known long before (see for example [P26] [FMSUW65]) with the model of the hydrogen atom [P26] as a classical example. Such systems are also known as superintegrable systems [FMSUW65] and as systems with non-commutative integrability [F88].

We will say that a subalgebra A of the algebra of functions on a smooth manifold $\mathcal{M}$ is generated by functions $f_1, \dots, f_n$ if for each function $f \in A$ the form $df \wedge \wedge f_1 \wedge \dots \wedge \mathfrak{h}_n$.

Assume that we have the following structure on a real symplectic manifold $(\mathcal{M}_{2n}, \omega)$.

- $2n-k$ independent functions $J_1, \dots, J_{2n-k}$ generating Poisson subalgebra $C_J(\mathcal{M})$ in $C(\mathcal{M})$.
- k independent functions $I_1, \dots, I_k$ generating Poisson center of the Poisson subalgebra $C_J(\mathcal{M})$.

Let $H \in C(\mathcal{M})$ be a function which Poisson commute with $J_1, \dots, J_{2n-k}$:

$$\{H, J_i\} = 0, \quad i = 1, \dots, 2n-k$$

We will say that the level surface $\mathcal{M}(c_1, \dots, c_{2n-k}) = \{x \in \mathcal{M} | J_i(x) = c_i\}$ of functions J_i is called generic relative to functions $I_1, \dots, I_n$ if the form $dI_1 \wedge \dots \wedge dI_k$ does not vanish identically on it. Then the following is true [N72]:

Theorem 5 *1. Flow lines of H are parallel to level surfaces of J_i.*

2. Each connected component of a generic level surface has canonical affine structure generated by the flow lines of $I_1, \dots, I_k$.

3. The flow lines of H are linear in this affine structure.

When $k = n$ this theorem reduces to the Liouville integrability [Arn89].

When $\mathcal{M}_2 n = N_{2k} \times \tilde{N}_{2n-2k}$ and functions $I_1, \dots, I_k$ are constant along $\tilde{N}_{2n-2k}$ the degenerate integrability of a system with commuting integrals $I_1, \dots, I_k$ is equivalent to the Liouville integrability of the system on N_k with integrals $I_i|_N$.

Because the theorem 5 generalizes the Liouville theorem to the case when the dimension of invariant tori is less the the dimension of $\mathcal{M}$ we will call these system degenerate integrable systems.

Geometrically, the structure described above means that we have two Poisson projections

$$\mathcal{M}_{2n} \xrightarrow{\psi} B^J_{2n-k} \xrightarrow{\pi} B^I_k \tag{17}$$

where B^J and B^I are Poisson manifolds (level surfaces of J_i and I_i respectively) and B^I has the trivial Poisson structure .

Degenerate integrable systems admit action-angle variables. Let us call the point $b \in B_I$ *regular* if the connected components of fibers of the preimage $\pi^{-1}(b)$ are Poisson submanifolds in B_J which consist of a single open dense symplectic leaf.

Let $a \in B^I$ be a regular point and D be an open neighborhood of a. Choose a generic point $c \in B_J$ which belongs to an open dense symplectic leaf of one of the connected components of $\pi^{-1}(b)$. Let U be a neighborhood of c. Choose the trivialization of π over D:

$$\pi^{-1}(D) \simeq \pi^{-1}(b) \times D \ .$$

Let $\tilde{U} \subset \pi^{-1}(D)$ be a neighborhood of c such that with respect to this trivialization,

$$\tilde{U} \simeq U \times D \ .$$

Together with the choice of the trivialization of π over W this gives the isomorphism

$$f : \psi^{-1}(\tilde{U}) \simeq \psi^{-1}(c) \times U \times D \ .$$

The functions $I_1, \ldots, I_k$ give a local coordinate system on D. Their Hamiltonian flows generate k independent Hamiltonian vector fields on ψ^{-1}. Define affine coordinates $\phi_1, \ldots, \phi_k$ on this level surface as natural coordinates along these vector fields.

Theorem 6 *Assume that $\psi^{-1}(c)$ is compact. Then there exist a trivialization $f : \psi^{-1}(\tilde{U}) \simeq \psi^{-1}(c) \times U \times D$ such that the symplectic form ω on $\mathcal{M}$ has the form*

$$\omega = f^* \left(\sum_{i=1}^{k} dI_i \wedge d\varphi_i + \pi^*(\omega_{\pi^{-1}(b)}) \right)$$

where $\omega_{\pi^{-1}(b)}$ is the symplectic form on the open dense symplectic leaf of the connected component of $\pi^{-1}(b)$ which contains c.

The coordinates ϕ_i, I_i are called action-angle variable for degenerate integrable systems.

One can replace real smooth manifolds by complex manifolds (complex algebraic) and Poisson structures by complex holomorphic (complex algebraic) structures. In this paper we assume that $\mathcal{M}$ is an affine algebraic variety. Poisson structure on $\mathcal{M}$ is determined by the structure of Poisson algebra on the ring of functions.

A degenerate integrable system on an algebraic symplectic manifold $\mathcal{M}$ consists of Poisson subalgebra J in the algebra of functions on $\mathcal{M}$ with $dim(Spec(J)) = 2n - k$ and if $Z(J)$ is the Poisson center of J, then $dim(Spec(Z(J))) = k$. Here $Spec(A)$ is the spectrum of primitive ideals of A.

5 Degenerate integrability of characteristic systems on standard simple Lie groups with Poisson Lie structure

5.1 Poisson structure on $G \times G//Ad_{G^*}$

5.1.1 The map ψ

Let M be an affine algebraic variety and let G be an algebraic Lie groups acting on it. Denote by $C_G(M)$ the algebra of G-invariant functions on M. We will use notation $M//G$ for the affine variety which is the categorical quotient, $M//G = Spec(C_G(M))$. Assume that M is Poisson (the ring of functions on M is a Poisson algebra). If M has a Poisson structure, G is a Poisson Lie group and the action is Poisson, then the algebra $C_G(M)$ is a Poisson subalgebra in $C(G)$ and therefore $M//G = Spec(C_G(M))$ has natural Poisson structure on it.

Theorem 7 *Let G be an algberaic Poisson Lie group. The projection $D(G) \to D(G)//Ad_{G^*}$ is a Poisson map.*

Proof. Similar to the previous section consider two functions f and g on $D(G)$ which are invariant with respect to the adjoint action of the subgroup G^*. Let $x \in D(G)$ and $b \in G^*$, then

$$\begin{aligned}
\{f,g\}(bxb^{-1}) &= \ < r_D, d_+f \otimes d_-g - d'_+f \otimes d'_-g > (bxb^{-1}) && (18)\\
&= \ < (Ad_b \otimes Ad_b)(r_D), (df - d'f) \otimes d_-g > (x) && (19)\\
&= \ < (Ad^*_b \otimes Ad_b)(r_D), (d_+f - d'_+f) \otimes d_-g > (x) && (20)\\
&+ \ < (Ad^*_b \otimes Ad_b)(r_D), (d_-f - d'_-f) \otimes d_-g > (x) && (21)\\
&= \ < (Ad^*_b \otimes Ad_b)(r_D), (d_+f - d'_+f) \otimes d_-g > (x) && (22)
\end{aligned}$$

Here d_+f is the differential "in the G-direction in $D(G)$", d_-f is the differential "in G^*-direction in $D(G)$" and r_D is the r-matrix for the double $D(\mathfrak{g})$ which is invariant with respect to $Ad^* \otimes Ad$ action of G^*. Thus, the Poisson bracket of two G^*-invariant functions is again G^*-invariant and this proves the theorem.

Define the variety $D(G)//Ad_{D(G)}$ again as the geometric quotient, i. e. as the spectrum of the $Ad_{D(G)}$-invariant functions on $D(G)$. Since $D(G) = G \times G$ and $G//Ad_G \simeq H/W$ we have

$$D(G)//Ad_{D(G)} \simeq H/W \times H/W$$

The natural imbedding $G^* \subset D(G)$ gives the map

$$D(G)//Ad_{G^*} \to D(G)//Ad_{D(G)}$$

and therefore the inclusion $C_{D(G)}(G) \subset C_{G^*}(G)$.

Proposition 5 *The map $D(G)//Ad_{G^*} \to D(G)//Ad_{D(G)}$ is Poisson.*

Proof. We should show that the subalgebra $C_{D(G)}(D(G)) \subset C_{G^*}(D(G))$ is in the center of the Poisson algebra $C_{G^*}(G)$. Let f be an $Ad_{D(G)}$-invariant function on $D(G)$ and g be Ad_{G^*}-invariant function on $D(G)$. Then $df = d'f,\ d_-g = d'_-g$ and therefore

$$\{f,g\}(x) =< r_D, (d_+f - d'_+f) \otimes d_-g >= 0$$

Thus, the pull-back of η gives central functions on $D(G)/Ad_{G^*}$. This proves the proposition.

Consider the composition map

ψ :

$$\psi : G \to D(G) \to D(G)//Ad_{G^*} \tag{23}$$

The map ψ is a composition of Poisson maps and therefore is a Poisson map itself. Now assume that G is a factorizable Poisson Lie group. In this case $D(G) = G \times G$.

Theorem 8 *1. $\psi^{-1}(\psi(g)) = Z(g) \cap G'$ where $Z(g)$ is the centralizer of g in G and G' is the subset of factorizable elements.*

2. The following diagram is commutative

$$\begin{array}{ccc} D(G)//Ad_{G^*} & \to & D(G)//Ad_{D(G)} = G//Ad_G \times G//Ad_G \\ \uparrow & & \uparrow \\ G & \to & G//Ad_G \end{array}$$

Here left vertical arrow is the map ψ and the right vertical arrow is the diagonal embedding.

Proof.

Now assume that G is a simple Lie group with a factorizable Poisson Lie structure. In this case $G//Ad_G \simeq H/W$ and we have a composition of Poisson projections

$$G \to \psi(G) \to H/W \tag{24}$$

Corollary 2 *If g simple element $dim(\psi^{-1}(\psi(g))) = r$.*

5.1.2 The map β

The following lemma is a combination of well known facts(see [Dr87] and [STS85] for example).

Lemma 6 *1. The subset $[B^- \times B^+] = \{(b_-, b_+) \in B^- \times B^+ | [b_+]_0 = [b_-]_0\}$ is a Poisson submanifold in $B^- \times B^+$.*

2. The map

$$[B^- \times B^+] \to G, \ (b^-, b^+) \to b^+(b^-) \tag{25}$$

is Poisson. Its image is open dense in G and it is a covering map with the group of deck transformations isomorphic to $\Gamma = \{\varepsilon \in H | \varepsilon^2 = 1\}$.

3. The adjoint action $h : (b_-, b_+) \mapsto (h^{-1} b_- h, h b_+ h^{-1})$ of $j(H)$ on $[B^- \times B^+]$ is Poisson.

Notice that $[B^- \times B^+]$ can be naturally identified with the left coset $(B^- \times B^+)/j(H)$ as a Poisson manifolds.

Corollary 3 *The coset manifold $[B^- \times B^+]/Ad_{j(H)}$ carries natural Poisson structure.*

Proposition 6 *There is an isomorphism of Poisson manifolds*

$$[B^- \times B^+]/Ad_{j(H)} \simeq [N^- \times N^+]/Ad_{j(H)} \times H \tag{26}$$

This proposition follows directly from the explicit form of the Poisson brackets for B^+ (see Appendix 2).

We have a natural map $B^- \times B^+ \to G/Ad_{N^+} \times G/Ad_{N^-}$ acting as

$$(b_-, b_+) \mapsto (Ad_{N^+}(b_-), Ad_{N^-}(b_+)) \tag{27}$$

This map induces the map

$$\beta : [B^- \times B^+]/Ad_{j(H)} \to D(G)//Ad_{G^*} = (G//Ad_{N^+} \times G//Ad_{N^-})/Ad_{j(H)}. \tag{28}$$

Theorem 9 *The map β is Poisson.*

Proof. It follows from the theorem 7 that the map $G \times G \to (G \times G)//Ad_{G^*}$ is Poisson. Consider the composition map $B^- \times B^+ \to G \times G \to (G \times G)//Ad_{G^*}$. Each of these maps is Poisson, therefore the composition is also Poisson.

In Appendix 2 we explain why maps $B^\pm \to G/Ad_{N^\mp}$ are branched cover maps over their images. The third part of the theorem follows from this.

Let $I : [B^- \times B^+]/Ad_H \to H$ be the projection to the second factor in (26). It is a Poisson map.

Theorem 10 *1. Let $g \in G$ be a semisimple element (i.e. the elements which is conjugate to an element of H). Then $\psi(g) \subset \beta([B^- \times B^+]/Ad_{j(H)})$ (in other words, then there exist elements $n_\pm \in N^\pm$ and $b_\pm \in B^\pm$ such that $g = n_+ b_- n_+^{-1} = n_- b_+ n_-^{-1}$) .*

2. The following diagram is commutative

$$\begin{array}{ccc} D(G)//Ad_{G^*} & \longrightarrow & D(G)//Ad_{D(G)} = H/W \times H/W \\ \uparrow \beta & & \uparrow \\ [B^- \times B^+]/Ad_{j(H)} & \xrightarrow{\hat{I}} & H/W \end{array}$$

Here the map $\hat{I}$ is the composition of the map I and of the natural projection $H \to H/W$ and the right vertical map is the the diagonal embedding $H/W \to H/W \times H/W$.

Proof. The second part of the theorem is obvious.

Let us prove that ψ map semisimple elements into the image of β in $D(G)/Ad_{G^*}$. Assume that $x = ghg^{-1}$ where $h \in H$ and $g \in G$. Assume that $g \in B^+ w B^+$ for some $w \in W$. Let $\bar{w}$ be a representative of w in $N(H) \subset G$. Then $g = b_+ \bar{w} n_+$ for some $b_+ \in B^+$ and $n_+ \in N_w^+$ where N_w^+ is the subset of elements in N^+ which map into N^- after conjugation with $\bar{w}$. We have:

$$\begin{aligned} ghg^{-1} &= b_+ \bar{w} n_+ h n_+^{-1} \bar{w}^{-1} b_+^{-1} && (29) \\ &= b_+ w(h) (\bar{w} h^{-1} n_+ h n_+ \bar{w}^{-1}) b_+^{-1} && (30) \\ &= b_+ w(h) \tilde{n}_- b_+^{-1} && (31) \end{aligned}$$

This proves that each semisimple element can be written as $x = n_+ b_- n_+^{-1}$ for some $n_+ \in N^+$ and $b_- \in B^-$. Similarly one can prove that $x = n_- b_+ n_-^{-1}$ for some $n_- \in N^-$ and $b_+ \in B^+$. This proves the first part of theorem.

5.2 Symplectic leaves of $[B^- \times B^+]/Ad_{j(H)}$

Let $u \in W$ be an element of the Weyl group and $u = s_{i_1} \dots s_{i_\ell}$ be its reduced decomposition. The subset $|u| \subset \{1, \dots, r\}$ of numbers which appear in the sequence $\{i_1, \dots, i_r\}$ is called the support of u.

Denote by $H(u)$ the subgroup of H generated by 1-parametric subgroups corresponding to simple roots α_i with $i \in |u|$. We have the following decomposition of $B^- \times B^+$:

$$B^- \times B^+ = \sqcup_{u,v \in W} B_u^- \times B_v^+,$$

where $B_w^\pm = B^\pm \cap B^\mp w B^\mp$. This decomposition gives the decomposition of $[B^- \times B^+]$:

$$[B^- \times B^+] = \sqcup_{u,v \in W} [B^- \times B^+]_{v,u}$$

where

$$[B^- \times B^+]_{v,u} = \{(b^-, b^+) \in B_v^- \times B_u^+ \mid [b^+]_0 = [b^-]_0, [b^\pm]_0 \in H\}$$

Theorem 11 *1. The subsets $[B^- \times B^+]_{v,u}$ are Poisson submanifolds.*

2. The Poisson submanifold $[B^- \times B^+]_{v,u}$ is fibered over $\mathbb{C}^{d_{u,v}}$ where $d_{u,v} = dim(ker_{\mathfrak{h}^*}(uv^{-1} - id)$ with fibers being common level sets of functions $c_{u,v,t}(b^- b^+)$, where $t \in ker_\Lambda(uv^{-1} - 1) \subset \mathfrak{h}^*$. Symplectic leaves of $[B^- \times B^+]_{v,u}$ are irreducible components of these fibers.

Proof. The map $[B^- \times B^+] \to G$, $(b^-, b^+) \mapsto b^- b^+$ is Poisson. It is a cover map with the group of deck transformations $\Gamma = \{\varepsilon \in H \mid \varepsilon^2 = 1\}$ with the image which is open dense in G. Symplectic leaves of G are irreducible components of level surfaces of functions $c_{u,v,t}$. The intersection of each symplectic leaf of G with the image of this map is open dense in the symplectic leaf . Therefore symplectic leaves of $[B^- \times B^+]$ are irreducible components of preimages of open dense subsets of symplectic leaves of G. This proves the theorem.

Proposition 7 *The subgroup $(H(u)H(v))^\perp \subset H$ is the stabilizer of the adjoint action of $j(H)$ on $[B^- \times B^+]_{v,u}$.*

This follows from the factorization formulae for B_v^-, B_u^+ (see [FZ99]).

Proposition 8 *The adjoint action of $H(u,v) = H/(H(u)H(v))^\perp \simeq H(u)H(v)$ is transitive on $[B^- \times B^+]_{v,u}$ and is Hamiltonian. Corresponding Hamiltonian vector fields are generated by linear functions on the Lie algebra of $H(u,v)$.*

Transitivity of the adjoint action of $H(u,v)$ is obvious. The second part of the theorem follows from the r-matrix form of Poisson brackets on G.

For $t = \sum_{i=1}^r t_i \omega_i \in ker_\Lambda(uv^{-1} - id) \subset \mathfrak{h}^*$ define functions $[c_u \otimes c_v]_t$ on $[B^- \times B^+]$ as

$$[c_v \otimes c_u]_t(b_-, b_+) = \prod_{i=1}^r \Delta_{\omega_i, u^{-1}\omega_i}(b_+)^{t_i} \Delta_{v\omega_i, \omega_i}(b_-)^{u^{-1}(t)i} \tag{32}$$

These functions are $Ad_{j(H)}$-invariant and therefore define functions on $[B^- \times B^+]/Ad_{j(H)}$.

Corollary 4 *The set of orbits of adjoint action of $j(H)$ passing through $[B^- \times B^+]_{v,u}$ is isomorphic to $[B^- \times B^+]_{v,u}/Ad_{j(H(v,u))}$.*

Theorem 12 *1. The isomorphism (26) induces the isomorphism of Poisson varieties $[B^- \times B^+]_{v,u}/Ad_{j(H(u,v)} \simeq (N_v^- \times N_u^+)/Ad_{j(H(v,u))} \times H$. The Poisson structure on the last factor is trivial.*

2. *Functions $[c_u \otimes c_v]_t$ are constant the subspace $H(u,v)$ in the second factor.*

3. *Symplectic leaves of $(N_v^- \times N_u^+)/Ad_{j(H(v,u))}$ are common level sets of functions $[c_u \otimes c_v]_t$, where $t \in ker_\Lambda(uv^{-1} - 1) \subset \mathfrak{h}^*$.*

This theorem follows from the Hamiltonian reduction via moment map.

5.3 Symplectic leaves of $\psi(G) \subset D(G)//Ad_{j(G)}$

Let $\psi : G \to (G \times G)/Ad_{j(G^*)}$ and $\beta : [B^- \times B^+]/Ad_{j(H)} \to (G \times G)/Ad_{j(G^*)}$ be the maps defined in (23)(28).

For $u, v \in W$ and $t = \sum_{i=1}^r t_i\omega_i \in \mathfrak{h}^*$ define functions $\tilde{c}_{u,v,t}$ on $G \times G$ as

$$\tilde{c}_{u,v,t}(g_1, g_2) = \prod_{i=1}^r \Delta_{\omega_i, u^{-1}\omega_i}(g_1)^{t_i} \Delta_{v\omega_i,\omega_i}(g_2)^{u^{-1}(t)_i}$$

Here functions $\Delta_{u\omega_i, v\omega_i}$ are defined in (10).

Proposition 9 *1. Function $\tilde{c}_{u,v,t}$ are invariant with respect to the $Ad_{j(G^*)}$-action.*

2. *$c_{u,v,t} = \psi^*(\tilde{c}_{u,v,t})$.*

3. *$[c_u \otimes c_v]_t = \beta^*(\tilde{c}_{u,v,t})$ were functions $[c_u \otimes c_v]_t$ are defined in (32).*

Proof. Recall that functions Δ have the following property:

$$\Delta_{\omega_i, u^{-1}\omega_i}(\xi_-^{-1} g \xi_-) = [\xi_-]_0^{-\omega_i + u^{-1}\omega_i} \Delta_{\omega_i, u^{-1}\omega_i}(g)$$

$$\Delta_{v\omega_i,\omega_i}(\xi_+^{-1} g \xi_+) = [\xi_+]_0^{\omega_i - v\omega_i} \Delta_{v\omega_i,\omega_i}(g)$$

This proves the first part of the proposition. The second part follows from the definition of $c_{u,v,t}$. For $\beta^*(\tilde{c}_{u,v,t})$ we have:

$$\beta^*(\tilde{c}_{u,v,t})(b_-, b_+) = \prod_{i=1}^r \Delta_{\omega_i, u^{-1}\omega_i}(b_+)^{t_i} \Delta_{v\omega_i,\omega_i}(b_-)^{u^{-1}(t)_i} = [c_u \otimes c_v]_t(b_-, b_+)$$

This proves the third part.

Let $S_g^{u,v} \subset G$ be the symplectic leaf in G passing through g. Denote by $H^W(u,v)$ the set of W-orbits in H which intersect $H(u,v) \subset H$.

Theorem 13 *Let $g = n_+ b_- n_+^{-1} = n_- b_+ n_-^{-1} \in G^{u,v}$ be a semisimple element. Denote $[g] = Ad_{j(H)}(b_-, b_+) \in [B_u^- \times B_v^+]/Ad_{j(H)}$ and denote by $\Sigma_{[g]}^{u,v}$ the symplectic leaf in $[B_u^- \times B_v^+]/Ad_{j(H)}$ passing through $[g]$. Let $\mathcal{O}_g$ be the Ad_G-orbit passing through G. Then*

1. *if (b'_-, b'_+) is a different pair representing g then $\Sigma_{[g]}^{u,v} = \Sigma_{[g]'}^{u,v}$.*
2. *$\psi(S_g^{u,v} \cap \mathcal{O}_g) = \beta(\Sigma_{[g]}^{u,v})$.*
3. *$\pi \circ \psi(S_g^{u,v} \cap \mathcal{O}_g) = A_G(g) \in H^W(u,v)$, assuming the identification $G//Ad_G = H/W$.*

Proof. The symplectic leaf $S_g^{u,v}$ is the common level surface of functions $c_{u,v,t}$ which contains g. It follows from this fact and from the proposition 9 that the common level surface of function $\tilde{c}_{u,v,t}$ which contains $\psi(g)$ is a Poisson subvariety in $\psi(G)$. On the other hand $\psi(g) = \beta([g])$ and by similar reasoning the subvariety of the common level surface of $\tilde{c}$ passing through $\psi(g)$ which consists of ψ images of elements $g' = n'_+ b'_- n'^{-1}_+ = n'_- b'_+ n'^{-1}_-$ with $[b'_+]_0 = w[b_+]_0$ for some $w \in W$, is $\beta(\Sigma_{[g]}^{u,v})$.

Corollary 5 *If g is semisimple the submanifold $\psi(S_g^{u,v} \cap \mathcal{O}_g) = \beta(\Sigma_{[g]}^{u,v})$ is a symplectic leaf in $\psi(G)$.*

Corollary 6 *If $[h] \in H^W(u,v)$ is a generic orbit corresponding to the coset $Ad_G(g)$, then connected components of $\pi^{-1}([h])$ are symplectic leaves of $\psi(S_g^{u,v})$ of dimension $dim(S^{u,v}) - 2dim(H(u,v)$.*

5.4 Integrability of characteristic systems on G

In order to prove the integrability of the characteristic system on the symplectic leaf $S^{u,v} \subset G^{u,v}$ we should describe the system of projections with properites (17).

Let $S_g^{u,v}$ be a symplectic leaf in G through $g \in G^{u,v}$. The restriction of Poisson maps (24) to $S_g^{u,v} \subset G$ gives the composition map

$$S_g^{u,v} \xrightarrow{\psi} \psi(S_g^{u,v}) \xrightarrow{\pi} Ad_G(S_g^{u,v}) \simeq H^W(u,v) \tag{33}$$

where $Ad_G(S_g^{u,v})$ is the set Ad_G orbits intersecting $S_{u,v}$ and $H^W(u,v)$ is the set of W-orbits in H passing through $H(u,v)$.

Let $[h_g] \in H^W(u,v)$ be the element corresponding to the orbit $Ad_G(g)$. According to the corollary 6 connected components of $\pi^{-1}([h_g])$, $[h_g] \in H^W(u,v)$ are symplectic leaves of $\psi(S_g^{u,v})$ of dimension

$$\dim(\pi^{-1}([h_g])) = \dim(S_g^{u,v}) - 2\dim(H(u,v)) \ .$$

Lemma 7 *Let $g \in G^{u,v}$ be a semisimple element. Then $dim(\psi^{-1}(\psi(g)) = dim(H(u,v)$.*

Proof. Assume that $g.g' \in G^{u,v}$ are semisimple elements such that $\psi(g) = \psi(g')$ Then

$$g' = \tilde{b_{\pm}} g \tilde{b_{\pm}}^{-1}$$

for some $\tilde{b_{\pm}} \in B^{\pm}$ with $[\tilde{b_+}]_0 = [\tilde{b_{\pm}}]_0^{-1}$.

Since g, g' are semisimple we can represent them as

$$g = n_{\pm} b_{\mp} n_{\pm}^{-1}, \quad g' = n'_{\pm} b'_{\mp} {n'}_{\pm}^{-1}$$

where $b_- \in B_v^-$ and $b_+ \in B_u^+$. Then we have:

$$b'_- = \beta_+ b_- \beta_+^{-1}, \quad \beta_+ = {n'}_+^{-1} \tilde{b}_+ n_+,$$

$$b'_+ = \beta_- b_+ \beta_-^{-1}, \quad \beta_- = {n'}_-^{-1} \tilde{b}_- n_-.$$

Notice that $[\beta_{\pm}]_0 = [\tilde{b}_{\pm}]_0$ and therefore $[\beta_+]_0 = [\beta_-]_0^{-1}$. Let $\beta_{\pm} = h^{\pm 1} \nu_{\pm}$ where $\nu_{\pm} \in N^{\pm}$. For $\nu_{\pm}$ there is only discrete choice (determined by the action of the Weyl group on H). The subgroups $H(u)^{\vee}$ and $H(v)^{\vee}$ act trivially (via conjugation) on B_u^+ and on B_v^- respectively. Therefore for given semisimple $g \in G^{u,v}$ the variety of semisimple elements $g' \in G^{u,v}$ such that $\psi(g) = \psi(g')$ has dimension $dim(H(u,v)$ and therefore $dim(\psi^{-1}(\psi(g))) = dim(H(u,v)$.

This lemma together with previous results proves the following theorem.

Theorem 14 *Projection ψ in (33) has $dim(H(u,v)$-dimensional kernel, the image of π has the same dimension and connected components of π^{-1} of generic points are symplectic leaves in the image of ψ. Therefore a Hamiltonian system generated by an Ad_G-invariant function on $S_g^{u,v}$ is integrable.*

Remark 2 *When $H(u,v) \neq H$ (or, equivalently, when reduced decompositions of u and v contain all simple reflections) we say that the symplectic leaf $S^{u,v}$ is not of full rank. In this case it is a symplectic leaf of the full rank an appropriate semi-simple Poisson Lie subgroup in G.*

Remark 3 *Among symplectic leaves in G of full rank there are symplectic leaves corresponding Coxeter elements. They have dimension $2r$ and are integrable in the usual Liouville sense (the invariant tori have dimension r). Corresponding integrable systems have been studied in [HKKR]. They are deformations of Toda systems.*

5.5 Action-angle variables

We will say the element $x \in G$ of a simple, complex Lie group is generic if it is a conjugate to a generic element from the Cartan subgroup: $x = uhu^{-1}$. Let V_λ be a finite dimensional irreducible representation of G with the weight decomposition

$$V_\lambda = \oplus_{\mu \in \mathfrak{D}(\lambda)} V_\lambda(\mu).$$

For generic $x \in G$ denote by P_μ^λ the complete system of orthogonal projections on the eigenspace of x in V_λ :

$$x = \sum_\mu t_\mu P_\mu^\lambda, \qquad P_\mu^\lambda P_\nu^\lambda = P_\mu^\lambda \delta_{\mu,\nu}$$

where $t_\mu \in C^\times$. Since x is generic $P_\mu^\lambda = u Q_\mu^\lambda u^{-1}$, where Q_μ^λ is the projection to the subspace of $V_\lambda(\mu)$ in the weight decomposition of V_λ. For the same reason t_μ is the value of h on $V_\lambda(\mu)$.

Let H be an Ad-invariant function on G and $g_\pm(t,x)$ be the factorized components of $g(t,x) = \exp t\nabla H(x)$:

$$g_+(t,x) g_-(t,x)^{-1} = g(t,x) \tag{34}$$

This factorization exists for sufficiently small t.

Denote by v_λ the highest weight vector of V_λ and by $(\cdot,\cdot)$ the Cartan form, i.e the non degenerate bilinear form such that $(\omega(a)x, y) = (x, ay)$ where ω is the Cartan antiinvolution ($[\omega(a), \omega(b)] = -\omega([a,b])$):

$$\omega(e_i) = f_i,\ \omega(f_i) = e_i,\ \omega(h_i) = h_i$$

and we assume the normalization $(v_\lambda, v_\lambda) = 1$.

Introduce variables (functions on generic elements of G):

$$r_\mu^\lambda = (v_\lambda, P_\mu^\lambda v_\lambda)$$

Theorem 15 *Let $\{x(t)\}$ be the flow line of the Hamiltonian vector field generated by H, passing through c at t=0. Then*

$$r_\mu^\lambda(x(t)) = \frac{e^{-tX_\mu(x)} r_\mu^\lambda}{\sum_{\nu \in \mathfrak{D}_\lambda} e^{-tX_\nu(x)} r_\nu^\lambda} \tag{35}$$

Here $X_\mu(x)$ is the eigenvalue of $\nabla H(x)$ on P_μ^λ.

Proof: According to [STS85] we have:

$$x(t) = g_+(t,x)^{-1} x g_+(t,x)$$

where $g_\pm(x,t)$ are determined by (34). Therefore

$$r^\lambda_\mu(x(t)) = (v_\lambda, P^\lambda_\mu(x(t))v_\lambda) = (v_\lambda, g_+(t,x)^{-1}P^\lambda_\mu(x(t))g_+(t,x)v_\lambda)$$

On the other hand $g_\pm(t,x)$ are elements of the Borel subalgebras $B_\pm$ of G. Write $g_\pm(t,x)$ as

$$\begin{aligned} g_+(t,x) &= u_+(t,x)h(t,x) \\ g_-(t,x) &= u_-(t,x)h(t,x)^{-1} \end{aligned}$$

where $u_\pm$ belong to corresponding unipotent subgroups and h is in the Cartan subgroup.

According to the definition of $(\cdot,\cdot)$ we have:

$$r^\lambda_\mu(x(t)) = (\omega(g_-(t))^{-1}v_\lambda, g(t)^{-1}P^\lambda_\mu g_+(t)v_\lambda)$$

and therefore

$$r^\lambda_\mu(x,t) = h_\lambda(t)\bar{h}_\lambda(t)e^{tX_\mu(x)}r^\lambda_\mu \tag{36}$$

where

$$h(t)v_\lambda = h_\lambda(t)v_\lambda, \qquad \omega(h(t))v_\lambda = \bar{h}_\lambda(t)v_\lambda$$

On the other hand

$$(v_\lambda, g(t)^{-1}v_\lambda) = (v_\lambda, g_-(t)g_+(t)^{-1}v_\lambda) = h_\lambda(t)^{-1}\bar{h}_\lambda(t))^{-1} \tag{37}$$

and

$$(v_\lambda, g(t)^{-1}v_\lambda) = \sum_{\mu\in\mathfrak{D}_\lambda} e^{-X_\mu(x)t}r^\lambda_\mu$$

substituting this into (36) and (37) we obtain (35). q.e.d.

Let μ_1 and μ_2 be two weights in an irreducible representation V_λ. Consider

$$r_{\mu_1,\mu_2} = \frac{r^\lambda_{\mu_1}}{r^\lambda_{\mu_2}}$$

We have:

$$r_{\mu_1\mu_2}(t) = exp(t(X_{\mu_2}(x) - X_{\mu_1}(x)))r_{\mu_1\mu_2}$$

Therefore the logarithms of $r_{\mu_1\mu_2}$ are affine coordinates on invariant tori and therefore n independent variables of this type can serve as angle variables. The eigenvalues of x are action variables for the Toda system.

For example for SL_n we can choose $\lambda = \omega_1, \mu = \omega_1 - \alpha_1 - \cdots - \alpha_i$. This is equivalent to Moser's construction for Toda symplectic leaf for SL_n.

Remark 4 *Coxeter-Toda systems are characteristc systems on symplectic leaves corresponding to a pair of Coxeter elements of the Weyl group. In real totally positive case the action-angle variables are global coordinates on the phase space for such systems. It will be interesting to see if similar property holds for any symplectic leaf.*

One should note that this construction of action-angle variables is very similar to the one given by Kostant [K79] in linear case.

6 Conclusion

We proved that a Hamiltonian system on any symplectic leaf of a simple Poisson Lie group with the standard Poisson structure is integrable if the Hamiltonian is Ad_G-invariant. Liouville tori of such systems are intersections of dressing and adjoint orbits.

One of the most interesting next questions is to describe the spectrum of corresponding quantum systems. In case of Toda systems this involves Wittaker vectors and some other facts about principal unitary series of representations of split real form of G.

In our case the analongs of Wittaker vectors and of principal unitary series of representations for the split real forms of quantized universal enveloping algberas $U(\mathfrak{g})$ should play similar role.

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7 Appendix 1.

Here will prove some useful fact which was not used this paper.

Theorem 16 *Let $(D(G), p)$ be the double of Poisson Lie group G. Let $(D(G), p_*)$ be the Poisson structure on the manifold $D(G)$ induced by the factorization map. Then if f and g are Ad_{G^*} invariant functions on D,*

$$\{f, g\} = \{f, g\}_* \, .$$

Proof. The Poisson brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_*$ have the following form

$$\begin{aligned}
\{f,g\} &= \langle r, df\otimes dg\rangle - \langle r, d'f\otimes d'g\rangle = (d_+f,\ d_-g) - (d'_+f,\ d'_-g) \\
\{f,g\}_* &= +\langle r, df\wedge dg + d'f\wedge d'g\rangle + \langle r, d'f\otimes dg - d'g\otimes df\rangle \\
&= +\tfrac{1}{2}((d_+f,\ d_-g) + (d'_+f,\ d'_-g) - (d_+g,\ d_-f) - (d'_+g,\ d'_-f)) \\
&\quad -(d'_+f,\ d_-g) + (d'_+g,\ d_-f)
\end{aligned}$$

Now assume that f and g are Ad_{G^*}-invariant. Then $d_-f - d'_-f,\ \ d_-g = d'_-g$ and we have

$$\begin{aligned}
\{f,g\} &= (d_+f - d'_+f,\ d_-g) = \tfrac{1}{2}(d_+f - d'_+f, d_-g) - \tfrac{1}{2}(d_+g - d'_+g,\ d_-f) \\
\{f,g\}_* &= \tfrac{1}{2}((d_+f, d_-g) - (d'_+f, d_-g) - (d_+g, d_-f) + (d_+g, d'_-f) \\
&= -\tfrac{1}{2}(d_+f, d'_-g - d_-g) + \tfrac{1}{2}(d_+g, d'_-f - d_-f)
\end{aligned}$$

The theorem follows.

8 Appendix 2. The Poisson structure on $G//Ad_{B^-}$

8.1 Poisson brackets of Ad_{B^-}-invariant functions on G

Recall that the Poisson bracket on functions on the double $D(B_+) = G\times H$ of B_+ (with standard Poisson structure) has the following form

$$\{f,g\} = \langle r_0, d_0^+f\otimes d_0^-g\rangle - \langle r_0, d_0^{+'}f\otimes d_0^{-'}g\rangle + \langle r_1, d_+f\otimes d_-g\rangle - \langle r_q, d'_+f\otimes d'_-g\rangle\ .$$

Here we assume that decompositions $B_\pm = HN_\pm$ are fixed, together with imbeddings $B_\pm \hookrightarrow G\times H,\ \ b_\pm \mapsto (b_\pm, ([b]_0)^{\pm 1})$, where $[\ ]_0 : B_\pm \mapsto H$ are projections to the Cartan subgroup.

Elements r_0, r_1 are canonical elements in $\mathfrak{h}\otimes\mathfrak{h}$ and in $\mathfrak{n}_+\otimes\mathfrak{n}_-$ respectively

$$r_0 = \sum_i H_i^+\otimes (H^-)^i\ , \qquad r_1 = \sum_\alpha e_\alpha\otimes f_\alpha\ .$$

Here we assume that the first factor in $\mathfrak{h}\otimes\mathfrak{h}$ is the image of the Cartan subalgebra in B and the second in the image of the Cartan subalgebra in B_- under projections $[\ ]_0$.

Differentials $d_0^\pm$ are taken in the direction of Cartan subgroup $H\subset B^\pm$ imbedded in $G\times H$ via maps i and j respectively . Differentials $d_\pm$ are taken in the direction of $N_\pm\subset G\times H$. If one trivializes tangent bundle to G by identifying tangent spaces to G with $\mathfrak{g}$ we have

$$\begin{aligned}
\langle \xi_0, d_0^+f\rangle(g,h) &= \frac{d}{dt}f(e^{t\xi_0}g,\ e^{t\xi_0}h)|_{t=0}\ , \\
\langle \xi_0, d_0^{+'}f\rangle(g,h) &= \frac{d}{dt}f(ge^{t\xi_0},\ he^{t\xi_0})|_{t=0}\ , \\
\langle \xi,0, d_0^-f\rangle(g,h) &= \frac{d}{dt}f(e^{t\xi_0}g,\ e^{t\xi_0}h)|_{t=0}\ , \\
\langle \xi, d_0^{-'}f\rangle(g,h) &= \frac{d}{dt}f(ge^{t\xi_0},\ he^{-t\xi_0})|_{t=0}
\end{aligned}$$

and

$$\begin{aligned}\langle \xi_\pm, d_\pm f\rangle(g,h) &= \frac{d}{dt} f(e^{\xi_\pm t} g,\ h)|_{t=0}\ ,\\ \langle \xi_\pm, d'_\pm f\rangle(g,h) &= \frac{d}{dt} f(g e^{\xi_\pm t},\ h)|_{t=0}\ .\end{aligned}$$

Here $\xi_0 \in \mathfrak{h},\ \ \xi_\pm \in \mathfrak{n}_\pm$.

In other words,

$$\begin{aligned}\langle \xi, d_0^\pm f\rangle &= \langle \xi, d_0 f\rangle \pm \langle \xi, d_H f\rangle\\ \langle \xi, d_0^\pm f\rangle &= \langle \xi, d'_0 f\rangle \pm \langle \xi, d_H f\rangle\end{aligned}$$

where $d_0 f, d'_0 f$ are left and right differentials of f in the Cartan direction $H \subset G$, $d_H f$ is the differential in the direction of the second factor in $G \times H$. Thus, for the Poisson bracket we have

$$\begin{aligned}\{f,g\} = &\ \langle r_0, d_0 f \otimes d_0 g\rangle - \langle r_0, d'_0 f \otimes d_0 g\rangle\\ &+\langle r_0, d_H f \otimes d_0 g - d_0 f \otimes d_H g\rangle - \langle r_0, d_H f \otimes d'_0 g - d'_0 f \otimes d_H g\rangle\\ &+\langle r_1, d_+ f \otimes d_- g\rangle - \langle r_1, d'_+ f \otimes d'_- g\rangle\end{aligned}$$

Antisymmetrizing this bracket we obtain:

$$\begin{aligned}\{f,g\} = &\ \langle r_0, d_H f \otimes (d_0 g - d'_0 g) - (d_0 f - d'_0 f) \otimes d_H g\rangle\\ &+\tfrac{1}{2}\langle r_1, d_+ f \otimes d_- g - d_+ g \otimes d_- f - d'_+ f \otimes d'_- g + d'_+ g \otimes d'_- f\rangle\ .\end{aligned}$$

Let $C_{Ad_{B^-}}(D)$ be the algebra of Ad_{B^-}-invariant functions on D.

As it follows from the previous subsection the algebra $C_{Ad_{B^-}}(D)$ is finitely generated. Define the variety $D//Ad_{B^-} = Spec(C_{Ad_{B^-}}(D))$.

The adjoint action of B^- on D is trivial on H-component of $D = G \times H$. Thus,

$$D//Ad_{B^-} = G//Ad_{B^-} \times H \tag{38}$$

as a variety.

Lemma 8 *The formula (38) describes the Poisson variety $D(G)//Ad_{B^-}$ as the product of two Poisson varieties with the trivial Poisson structure on the H-factor.*

Indeed, if f and g are Ad_{B^-}-invariant functions on $G\times H$ we have $d_0 f = d'_0 f,\ d_- f = d'_- f$ and the same for g. Therefore for the Poisson bracket between f and g we have

$$\{f,g\} = \tfrac{1}{2}\langle r_1, (d_+ f - d'_+ f) \otimes d_- g\rangle - \tfrac{1}{2}\langle r_1, (d_+ g - d'_+ g) \otimes d_- f\rangle \tag{39}$$

This means that functions constant along G are central in the Poisson algebra which proves the lemma.

The map $B^+ \hookrightarrow D(B^+) \to D(B^+)//Ad_{B^-}$ is a composition of Poisson maps and therefore is Poisson. Projecting to the second factor in the (38) we have the Poisson projection:

$$B^+ \to G//Ad_{B^-}. \tag{40}$$

The projection $B^+ \to B^+/Ad_H$ is Poisson. This follows from the Ad_H-invariance of the standard Poisson structure on G. It is also clear that the diagram

$$\begin{array}{ccc} B^+ & \longrightarrow & G//Ad_{B^-} \\ & \searrow & \uparrow \pi \\ & & B^+/Ad_H \end{array}$$

is commutative. Therefore the map

$$B^+ \to G//Ad_{B^-}$$

is Poisson.

The image of this map is open dense in $G//Ad_{B^-}$ and the map is a finite branched cover. The number of branches over generic point is equal to $|W|$ and the Weyl group W acts naturally on the fibers.

8.2 Poisson structure on B^+/Ad_H

The Poisson structure on B^+/Ad_H can be described explicitly.

Theorem 17 *The Poisson bracket of two Ad_H-invariant functions on B^+ has the following form:*

$$\{f, g\}(x) = \langle (\mathrm{id} \otimes \mathrm{Ad}_{x^{-1}}(r_1), d_+ f \otimes \partial'_+ g \rangle (x) \tag{41}$$

Here $d_+ f$ is the left differential of f at the point $x \in B^+$ and $\partial'_+ g$ is the right differential of g projected on $\mathfrak{n}_+ \subset T_x B^+$.

Proof. This theorem can be derived as a pull-back of the Poisson structure on $G//Ad_{B^-}$ or from the restriction of the standard Poisson structure on G to Ad_H-invariant functions.

Let us first compute it as a pull-back.

Assume that $x \in G$ belongs to the image of (40), i.e. there exists an element $x_+ \in B^+$ and $n_- \in N^-$ such that $x = n_- x_+ n_-^{-1}$.

Lemma 9 *The value of the Poisson bracket of two* Ad_{B^-}*-functions on such element* $x = n_- x_+ n_-^{-1} \in G$ *has the following form*

$$\{f,g\}(x_+) = \{f,g\}(x_+) = \sum_i \langle \rho^i, d_+ f - d'_+ f\rangle (x_+) \langle \rho_i, d_- f\rangle (x_+) \tag{42}$$

Proof. For $\xi_+ \in \mathfrak{n}_+$ and $b_- \in B^-$ we have

$$\begin{aligned}
\langle \xi_+, d_+ f - d'_+ f\rangle (b_- x b_-^{-1}) &= \frac{d}{dt} f(e^{\xi_+ t} b_- x b_-^{-1}\ e^{-\xi_+ t})|_{t=0} \\
&= \frac{d}{dt} f(e^{(\xi_+^{b_-^{-1}})_+ t} x\ e^{-(\xi_+^{b_-^{-1}})_+ t})|_{t=0} \\
&= \langle (\xi_+^{b_-^{-1}})_+,\ d_+ f - d'_+ f\rangle (x)
\end{aligned}$$

Here $(\xi_+^{b_-^{-1}})_+ \in \mathfrak{n}_+$. On the other hand,

$$\langle \xi_-, d_- g\rangle (b_- x b_-^{-1}) = \frac{d}{dt} g(e^{t\xi_-} b_- x b_-^{-1})|_{t=0} = \frac{d}{dt} g(e^{\xi_-^{b_-^{-1}}} x)|_{t=0} = \langle \xi_-^{b_-^{-1}}, d_- g\rangle (x_+)\ .$$

The element $r_1 = \sum_i \rho^i \otimes \rho_i$ is $(\mathrm{Ad}^* \otimes \mathrm{Ad})_{N^-}$-invariant and is also invariant with respect to the diagonal action of H. Therefore

$$\begin{aligned}
\langle r_1, (d_+ f - d'_+ f) \otimes d_- g\rangle (b_- x b_-^{-1}) &= \sum_i \langle (((\rho^i)^{b_-^{-1}})_+, d_+ f - d'_+ f\rangle (x) \langle \rho_i^{b_-^{-1}}, d_- f\rangle (x_+) \\
&= \sum_i \langle \rho^i, d_+ f - d'_+ f\rangle (x) \langle \rho_i, d_- f\rangle (x)
\end{aligned}$$

This proves the lemma.

To prove the theorem we should verify that the Poisson bracket (42) between two Ad_{B^+}-invariant functions is given by (41).

For $\xi_+ \in \mathfrak{n}_+$ and $x_+ \in B^+$ we have:

$$\langle \xi_+, d_+ f - d'_+ f\rangle (x_+) = \langle \xi_+ - \mathrm{Ad}_{x_+}(\xi_+), d_+ f\rangle (x_+)\ .$$

For $\xi_- \in \mathfrak{n}_-$ and $x_+ \in B_+$ we have

$$\begin{aligned}
\langle \xi_-, d_- g\rangle (x_+) &= \frac{d}{dt} g(e^{t\xi_-} x_+)|_{t=0} \\
&= \frac{d}{dt} g(e^{t\eta_-} e^{t a_+} x_+ e^{-t\eta_-})|_{t=0} \\
&= \frac{d}{dt} g(e^{t a_+} x_+)|_{t=0} = \langle a_+, \partial_+ g\rangle (x_+)
\end{aligned}$$

Here $\eta_- \in \mathfrak{n}_-$ and $a_+ \in b_+$ satisfy the equation

$$\xi_- = \eta_- + a_+ - \mathrm{Ad}_{x_+}(\eta_-) \ .$$

This equation gives the equation for η_-:

$$\xi_- = \eta_- - (\mathrm{Ad}_{x_+}(\eta_-))_- \tag{43}$$

and for a_+ we have

$$a_+ = (\mathrm{Ad}_{x_+}(\eta_-))_+$$

Here $(\mathrm{Ad}_{x_+}(\eta_-))_+$ is the $\mathfrak{b}_+$ part of $\mathrm{Ad}_{x_+}(\eta_-)$. Thus,

$$\langle \xi_-, d_-g\rangle(x_+) = \langle (\mathrm{Ad}_{x_+}(\eta_-))_+,\ \partial_+ g\rangle(x_+)$$

where $\eta_- \in \mathfrak{n}_-$ is the solution to (43). Here $\partial_+ f$ is the differential of f "in the direction of $B^+ \subset G$.

For the value of the Poisson bracket of two Ad_{B^-}-invariant functions on the element $x = n_- x n_-^{-1} \in G$ we have ,

$$\begin{aligned} \{f,g\}(x) = \{f,g\}(x_+) &= \langle r_1 - (\mathrm{Ad}_{x_+} \otimes \mathrm{id})(r_1), d_+f \otimes d_-g\rangle(x_+) \\ &= \langle \rho^i - \mathrm{Ad}_{x_+}(\rho^i), d_+f\rangle(x_+)\langle \mathrm{Ad}_{x_+}(\sigma_i), \partial_+ g\rangle(x_+) \end{aligned}$$

Here σ_i is the solution to

$$\rho_i = \sigma_i - (\mathrm{Ad}_{x_+}(\sigma_i))_- \ .$$

Because r_1 is the canonical element in $\mathfrak{n}_+ \otimes \mathfrak{n}^-$ (assuming we fixed an isomorphism $\mathfrak{n}^- \cong (\mathfrak{n}_+)^*$ by the choice of Killing form) we have:

$$\mathrm{Ad}_{x_+}(\rho^i) \otimes \rho_i = \rho^i \otimes (\mathrm{Ad}_{x_+^{-1}}(\rho^i))_-$$

and therefore

$$(\rho^i - \mathrm{Ad}_{x_+}(\rho^i)) \otimes \sigma_i = -\rho^i \otimes (\mathrm{Ad}_{x_+^{-1}}(\rho^i))_- \ .$$

Thus,

$$\{f,g\}(x_+) = -\langle \rho^i, d_+f\rangle(x_+)\langle \mathrm{Ad}_{x_+}((\mathrm{Ad}_{x_+^{-1}}(\rho^i))_-), \partial_+ g\rangle(x_+) \ ,$$

On the other hand,

$$\begin{aligned} &\langle \mathrm{Ad}_{x_+}(\mathrm{Ad}_{x_+^{-1}}(\rho^i)_-), \partial_+ g\rangle(x_+) \\ &\quad = \langle \rho_i - \mathrm{Ad}_{x_+}(\mathrm{Ad}_{x_+^{-1}}(\rho^i))_+), \partial_+ g\rangle(x_+) \\ &\quad = -\langle \mathrm{Ad}_{x_+^{-1}}(\rho^i)_+, \partial'_+ g\rangle(x_+) \end{aligned}$$

The theorem is proved.

8.3 The second proof of the theorem 17

Let $G//N^-$ be the categorical quotient for the right action of N^- on G. The following is well known.

Theorem 18 *The map $G \to G//N^-$ is Poisson.*

Proof. Let f and g be two functions on G invariant with respect to the right action of N^-. We have $d'_- g = 0$ and

$$< r_0, d_0' f \otimes d_0' g > (xn_-) = \sum_{i=1}^{r} \frac{d^2}{dsdt} f(xn_- e^{th_i}) g(xn_- e^{sh^i}) =< r_0, d_0' f \otimes d_0' g > (x)$$

For the Poisson brackets of two such functions we have:

$$\{f, g\} = \langle r_0, d_0 f \otimes d_0 g - d_0' f \otimes d_0' g\rangle + \tfrac{1}{2}\langle r_1, d_+ f \otimes d_- g\rangle(x_+) - \tfrac{1}{2}\langle r_1, d_+ g \otimes d_- f\rangle$$

Here the first and the second term are invariant with respect to the left action of N^- the invariance of the second term is shown above and the forth term vanishes. Thus, the Poisson bracket of two invariant functions again invariant. Therefore the map is Poisson.

Theorem 19 *1. The map $\phi : B^+ \hookrightarrow G \to G//N^-$ is Poisson for the standard Poisson structure on G and and is a local isomorphism (an isomorphism in a neighborhood of the identity in B^+) .*

2. Let f and g be two left N^--invariant functions on G, then the pull-back of ϕ gives the following Poisson brackets

$$\begin{aligned} \{f, g\} &= \langle r_0, d_0 f \otimes d_0 g - d_0' f \otimes d_0' g\rangle \\ &\quad + \tfrac{1}{2}\langle(\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ f \otimes \partial_+ g\rangle - \tfrac{1}{2}\langle(\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ g \otimes \partial_+ f\rangle \\ &= \langle r_0, d_0 f \otimes d_0 g - d_0' f \otimes d_0' g\rangle + \langle(\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ f \otimes \partial_+ g\rangle \end{aligned}$$

This Poisson bracket is the standard Poisson structure on B^+.

Proof. The map ϕ is Poisson since the since the left action of N^- on G is admissible (in a sense of [STS85]). It is also clear that it is an isomorphism in a neighborhood of 1.

For the Poisson bracket on functions on G we have:

$$\{f, g\} = \langle r_0, d_0 f \otimes d_0 g - d_0' f \otimes d_0' g\rangle + \tfrac{1}{2}\langle r_1, d_+ f \otimes d_- g\rangle(x_+) - \tfrac{1}{2}\langle r_1, d_+ g \otimes d_- f\rangle$$

For $\xi \in \mathfrak{n}^-$ we have

$$\langle \xi, d_- f\rangle(x) = \frac{d}{dt} f(e^{t\xi} x_+)|_{t=0} = \frac{d}{dt} f(x_+ e^{t(\xi^{x_+^{-1}})_+})|_{t=0}$$

Here we used decomposition $\eta = \eta_+ + \eta_-$ where $\eta = \mathfrak{g}$, $\eta_+ \in \mathfrak{b}_+$, $\eta_- \in \mathfrak{n}^-$. Thus,

$$\begin{aligned} \{f, g\} &= \langle r_0, d_0 f \otimes d_0 g - d_0' f \otimes d_0' g \rangle \\ &\quad + \tfrac{1}{2} \langle (\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ f \otimes \partial_+' g \rangle \\ &\quad - \tfrac{1}{2} \langle (\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ g \otimes \partial_+' f \rangle \ . \end{aligned}$$

Corollary 7 *If functions f and g are Ad_H-invariant,*

$$\{f, g\} = \langle (\mathrm{id} \otimes \mathrm{Ad}_{x_+^{-1}})(r_1), d_+ f \otimes \partial_+ g \rangle$$

This gives the second proof of the formula (41).